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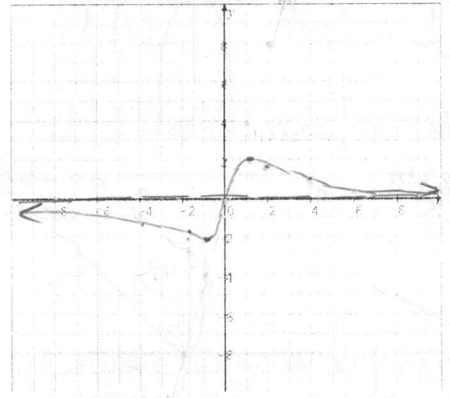
Math 12 Enriched: Section 3.5 Graphing Rational Functions and Applications

1. Given each equation below, graph it on the grid provided. Label all max. & min. points:

a) $y=\dfrac{4x}{x^2+1}$

$x^2+1\,\overline{)\,4x}$ with 0 above

Quotient = 0



Asymptotes: NO V.A. ; $y \neq 0$

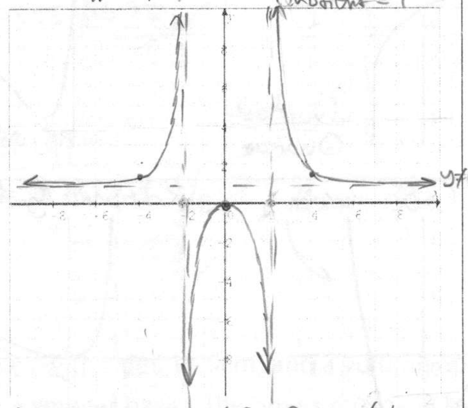
D: $x \in \mathbb{R}$ R: $-2 \le y \le 2$

b) $y=\dfrac{x^2}{x^2-4}$

$x^2-4\,\overline{)\,x^2}$ with 1 above

$-(x^2-4)$

Quotient = 1



Asymptotes: $x \neq 2, -2$; $y \neq 1$

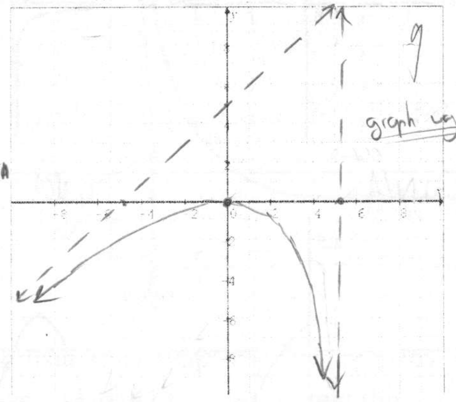
D: $x \in \mathbb{R}$; $x \neq \pm 2$ R: $y \le 0$ or $y > 1$

c) $y=\dfrac{x^2}{x-5}$

$x-5\,\overline{)\,x^2}$ with $x+5$ above

$-(x^2-5x)$

$5x$

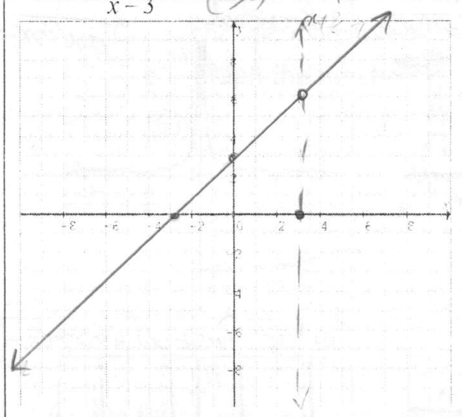


graph up above

Asymptotes: $x \neq 5$; $y \neq x+5$

D: $x \in \mathbb{R}$; $x \neq 5$ R: $y \le 0$ or $y > 1$

d) $y=\dfrac{x^2-9}{x-3}=\dfrac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}=\dfrac{x+3}{1}$ quotient $= x+3$

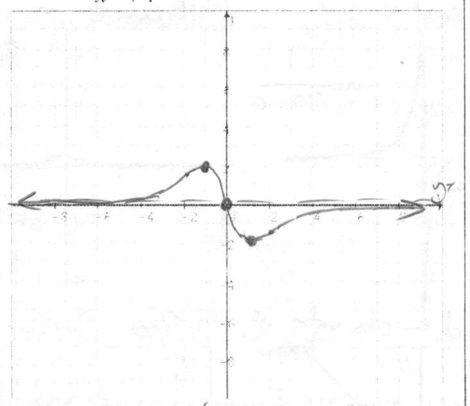


Asymptotes: $x \neq 3$;

D: $x \in \mathbb{R}$; $x \neq 3$ R: $y \in \mathbb{R}$

e) $y=\dfrac{-4x}{x^2+1}$

$x^2+1)\,-4x$



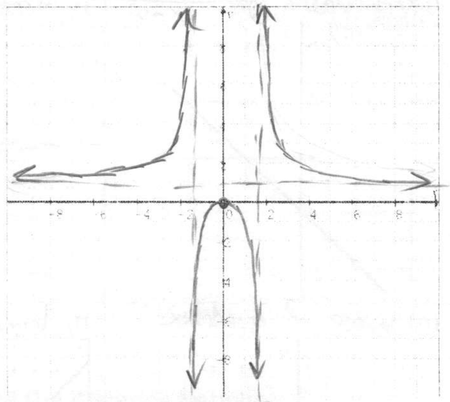
Asymptotes: NO V.A. ; $y \neq 0$

D:

R:

f) $y=\dfrac{x^2}{x^2-2}$

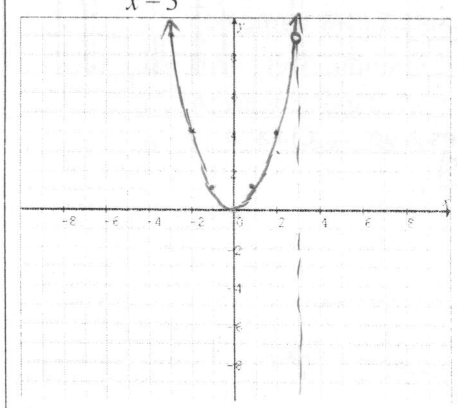
$x^2-2)\,x^2$ with 1 above



Asymptotes: $x \neq \pm\sqrt{2}$; $y \neq 1$

D: $x \in \mathbb{R}$; $x \neq \pm\sqrt{2}$ R: $y \le 0 \cup y > 1$

g) $y=\dfrac{x^3-3x^2}{x-3}=\dfrac{x^2(x-3)}{x-3}$



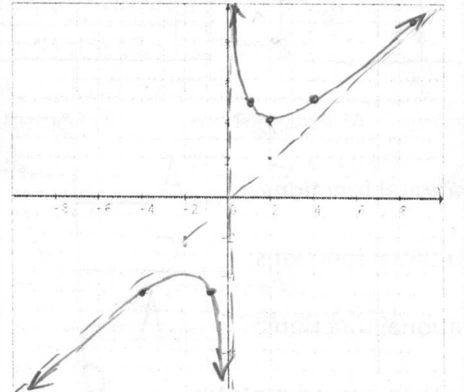
Asymptotes: $x \neq 3$

D: $x \in \mathbb{R}$; $x \neq 3$ R: $y \ge 0$

h) $y=\dfrac{x^2+4}{x}$

$x\,\overline{)\,x^2+4}$ with x above

x^2



Asymptotes: $x \neq 0$; $y \neq x$

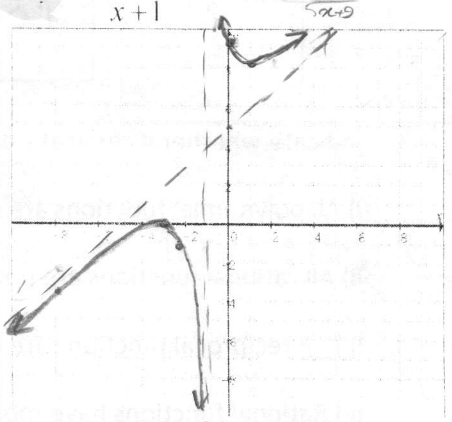
D: $x \in \mathbb{R}$; $x \neq 0$ R: $y \ge 4 \cup y \le -4$

i) $y=\dfrac{(x+3)^2}{x+1}$

$x+1\,\overline{)\,x^2+6x+9}$ with $x+5$ above

$-(x^2+x)$

$5x+9$

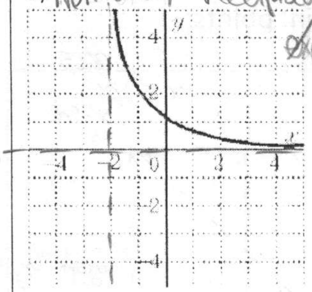


Asymptotes: $x \neq -1$; $y \neq x+5$

D: $x \in \mathbb{R}$; $x \neq -1$ R: $y \le 0 \cup y \ge 8$

2. Indicate which of the following graphs are rational functions only. If not, indicate what kind of function it is. If it is, indicate the general formula of the function and draw all the asymptotes:

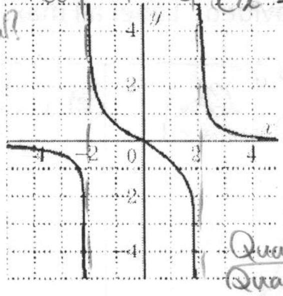
a) Non-rational – Reciprocal / exponential?



$y \neq 0$

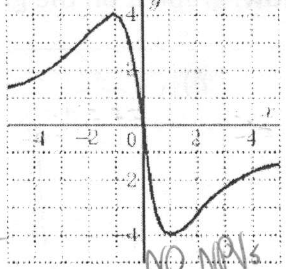
$x \neq -2$

b) Rational – $y=\frac{ax^2+bx}{cx^2-d}$



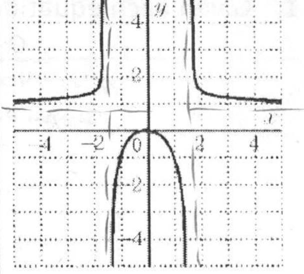
$\frac{\text{Quadratic}}{\text{Quadratic}}$

c) Rational $y=\frac{ax^2+bx}{cx^2+d}$

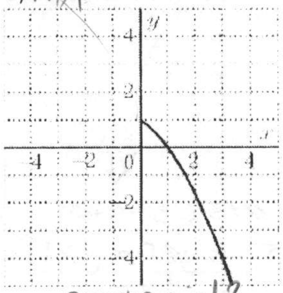


NO NPVs

d) Rational – $y=\frac{ax^2+bx}{cx+d}$

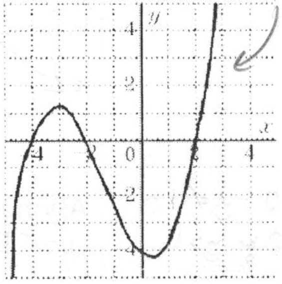


e) N/A

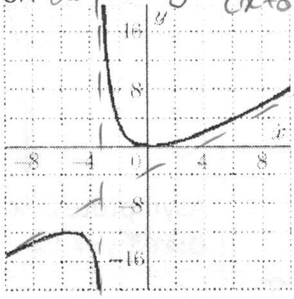


exponential?

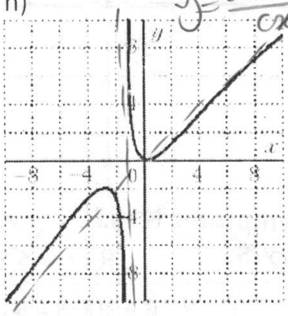
f) N/A Cubic function



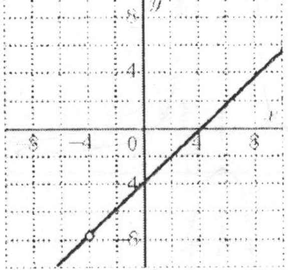
g) Rational $y=\frac{ax^2+bx}{cx+d}$



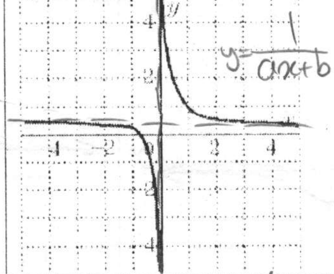
h) $y=\frac{ax^2+bx}{cx+d}$



i) Rational $y=\frac{(x-a)(x-b)}{(x-b)}$

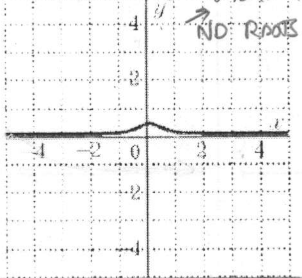


j) Rational

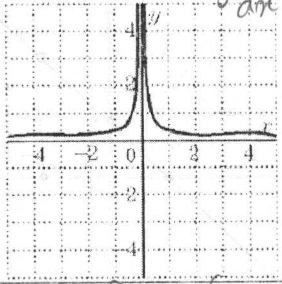


$y=\frac{1}{ax+b}$

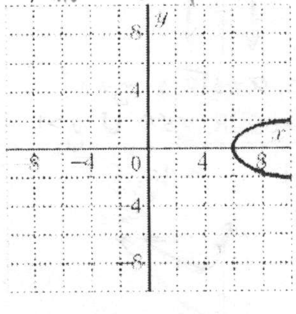
k) Rational $y=\frac{1}{ax^2+bx+c}$ → NO ROOTS



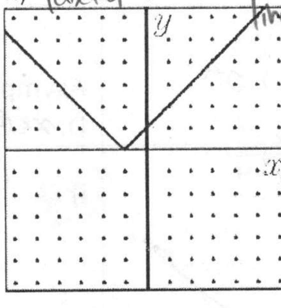
L) Rational $y=\frac{1}{ax^2+bx+c}$



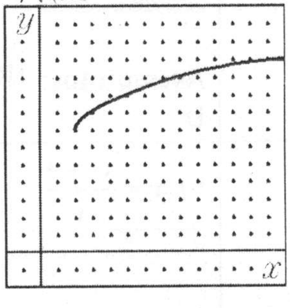
m) inverse of quadratic



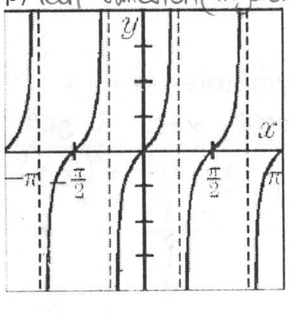
n) $|ax+b|$ ← abs. value of linear



0) Root function



p) tan function (trigonometry)



3. Indicate whether if the statement is A) Always True B) Sometimes True C) False

i) All polynomial functions are rational functions: A $\frac{P(x)}{1}$

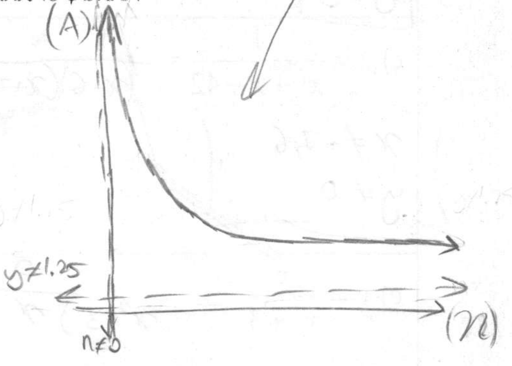
ii) All rational functions are polynomial functions: C

iii) All reciprocal functions are rational functions: A

iv) Rational functions have more than one asymptotes: B

4. The average cost "A" dollars, of printing the school agenda is given by the equation: $A = \frac{2500 + 1.25n}{n}$, where "n" is the number printed.
   a. Graph the function $0 \le n \le 900$
   b. Determine the average cost when 500 agendas are printed
   c. Determine the number of agendas are printed when the average cost is \$8.00?

a) $\begin{array}{r} 1.25 \\ n\overline{)2500+1.25n} \end{array}$



b) $A = \frac{2500+1.25(500)}{500} = 5+1.25 = \boxed{\$6.25}$

c) $8 = \frac{2500+1.25n}{n} \Rightarrow 6.75n = 2500$

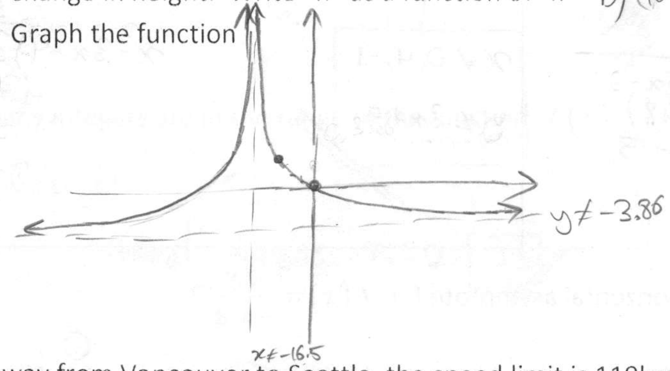
$\boxed{n = 370.37}$

5. A packaging company makes boxes with sides 16.5cm, and a volume approximately 1050cm^3. The company plans to redesign the boxes with a smaller base. The boxes must still have a square base and contain the same volume.
   a. Calculate the height of the box
   b. Let "x" centimeters represent the change in the length of the base. Let "h" centimeters represent the change in height. Write "h" as a function of "x"
   c. Graph the function

a) $h = \frac{1050}{(16.5)^2} = \boxed{3.8567...}$

b) $(16.5+x)^2(3.86+h) = 1050 \Rightarrow \boxed{h(x) = \frac{1050}{(16.5+x)^2} - 3.86}$

c)



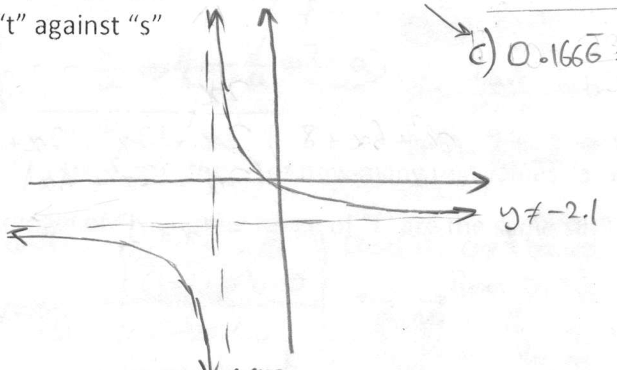
6. On the way from Vancouver to Seattle, the speed limit is 110km/h. Since the distance between the cities is approximately 230km, a trip between the two cities is about 2.1hours (Excluding border wait time). Cars travelling faster can reach their destination within 2.1 hours.
   a. Let "s" represent the change in speed compared with 110km/h. Let "t" represent the change in time compared with 2.1hrs. Write "t" as a function of "s"
   b. How much time will you save driving at 125km/h?
   c. At what speed does it take to save 10min?
   d. Graph "t" against "s"

a) $\frac{230}{110+s} = 2.1+t \Rightarrow t(s) = \frac{230}{110+s} - 2.1$

b) $t(15) = \frac{230}{125} - 2.1 = -0.26$ hours $= \boxed{15.6 \text{ mins}}$

c) $0.166\overline{6} = \frac{230}{110+s} - 2.1 \Rightarrow s = \frac{230}{2.2\overline{6}} - 110 = -8.529$

$110 + 8.529 = \boxed{118.53}$ km/h



7. Find all the vertical and horizontal asymptotes for each of the following rational functions:

a) $y=\frac{1}{x+3}$ $x\neq -3$ $y\neq 0$	b) $y=\frac{3x}{x+4}$ 3 $x+4\,\overline{)\,3x}$ $-(3x+12)$ -12 $x\neq -4$ $y\neq 3$
c) $y=\frac{1}{x^2+x-42}=\frac{1}{(x-6)(x+7)}$ $x\neq -7,6$ $y\neq 0$	d) $y=\frac{x^2}{x^2-16}=\frac{x^2}{(x-4)(x+4)}$ 1 $x^2-16\,\overline{)\,x^2}$ $x\neq \pm 4$ $y\neq 1$
e) $y=\frac{x}{x^2+3}$ 0 $x^2+3\,\overline{)\,x}$ NO V.A. $y\neq 0$.	f) $y=\frac{x^3}{6x^2-x-2}=\frac{x^3}{(2x+1)(3x-2)}$ $\frac{2}{3}$ $\frac{1}{-2}$ $\frac{1}{6}x+\frac{1}{36}$ $6x^2-x-2\,\overline{)\,x^3}$ $-(x^3-\frac{1}{6}x^2-\frac{1}{3}x)$ $\frac{1}{6}x^2-\frac{1}{3}x$ $x\neq -\frac{1}{2},\frac{2}{3}$ $y\neq \frac{1}{6}x+\frac{1}{36}$
$\frac{1}{2}$ $\frac{-1}{3}$ g) $y=\frac{2x^2+5x-3}{9-x^2}=\frac{(2x-1)(x+3)}{(3-x)(3+x)}$ $x\neq \pm 3$ $y\neq -2$ -2 $9-x^2\,\overline{)\,2x^2+5x-3}$ $-(2x^2-18)$ $5x+15$	h) $y=\frac{3x^4-27x^2}{x^3-3x^2-4x}=\frac{3x^2(x+3)(x-3)}{x(x-4)(x+1)}$ 1 -4 $3x+9$ $x^2-3x-4\,\overline{)\,3x^3-27x}$ $-(3x^3-9x^2-12x)$ $9x^2-15x$ $x\neq 0,4,-1$ $y\neq 3x+9$

8. What are the vertical asymptote and horizontal asymptote for $f(x)=\frac{\sqrt{x}}{x+4}$?

V.A: $x\neq -4$

H.A: $y\neq 0$

0

$x+4\,\overline{)\,x^{\frac{1}{2}}}$

0

9. Find all the vertical asymptotes for $f(x)=\frac{2x^3+12x^2+22x+12}{x^2+6x+8}=\frac{2x^3+12x^2+22x+12}{(x+2)(x+4)}$

2 4

V.A: $x\neq -2,-4$

H.A: $y\neq 2x$

$2x$

$x^2+6x+8\,\overline{)\,2x^3+12x^2+12x+12}$

$-(2x^3+12x^2+16x)$

$-4x+12$

10. Consider the function: $f(x)=\dfrac{x^2+x-6}{\underset{2}{2}x^2+7x+\underset{3}{3}}$. If "h" is the number of horizontal, "v" is the number of vertical, and "s" the number of slant asymptotes, what is the ordered triple (h,v,s)?

$f(x)=\dfrac{(x-2)\cancel{(x+3)}}{\cancel{(x+3)}(2x+1)}$ $\qquad x\neq -3,-\frac{1}{2}$

$\boxed{(h,v,s)=(1,2,0)}$ //

Diff. between numerator & denominator is a power of 0, so we have a perfectly horizontal assymptote & no slanted asymptote.

11. How many vertical asymptotes does $f(x)=\dfrac{4}{x^2+1}$ have?

$\boxed{\text{None!}}$ //

x^2+1 has no roots!!!

12. The graph of $y=\dfrac{3x^3+x^2+4}{x^2-48}$ in the cartesian plane has asymptotes $x=a$, $x=b$, and $y=cx+d$. Evaluate the value of $a\times b\times c\times d$.

V.A.: $x\neq\pm\sqrt{48}\Rightarrow x\neq\pm4\sqrt{3}$

H.A.: $y\neq 3x+1$

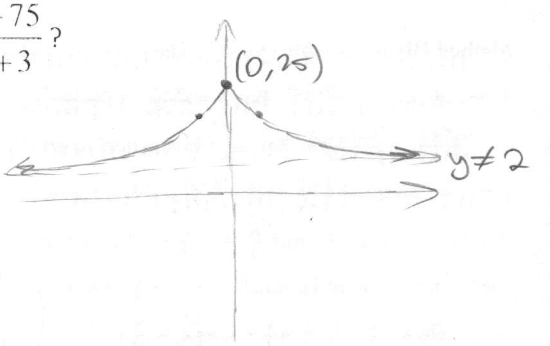
$$\begin{array}{r} 3x+1 \\ x^2-48\overline{)\,3x^3+x^2+4} \\ -(3x^3-144x) \\ \hline x^2+144x+4 \end{array}$$

$(4\sqrt{3})(4\sqrt{3})(3)(1)=\boxed{144}$ //

13. How many integers are in the range of the function $f(x)=\dfrac{4x^2+75}{2x^2+3}$?

Range: $2<y\leq25$

total integers: $25-2=\boxed{23}$ //



14. The function "f" is defined by: $f(x)=\dfrac{ax+b}{cx+d}$ where a, b, c, and d are non-zero real integers and have the properties: $f(19)=19$, $f(97)=97$, and $f(f(x))=x$ for all values of "x" except $\dfrac{-d}{c}$. Find the unique number that is not in the range of "f" (AIME)

① This unique # is the slanted assymptote, aka the quotient when $(ax+b)\div(cx+d)$. quotient $=\frac{a}{c}$, so we shall solve for $\frac{a}{c}$.

② $f\left(f\left(-\frac{b}{a}\right)\right)=\dfrac{a\left(\frac{a(-\frac{b}{c})+b}{c(-\frac{b}{c})+d}\right)+b}{c\left(\frac{a(-\frac{b}{c}+b)}{c(-\frac{b}{c}+d)}\right)+d}=-\dfrac{b}{a}\Rightarrow\dfrac{\cancel{b}}{d}=-\dfrac{\cancel{b}}{a}\Rightarrow d=-a$

③ $f(x)=\dfrac{ax+b}{cx-a}$

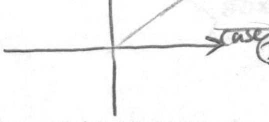
$f(19)=\dfrac{19a+b}{19c-a}=19\Rightarrow19^2c=2\cdot19a+b$

$f(97)=\dfrac{97a+b}{97c-a}\Rightarrow97^2c=2\cdot97a+b$

subtract $c(97^2-19^2)=2a(97-19)$

$\dfrac{a}{c}=\dfrac{97^2-19^2}{97-19}=\dfrac{(97-19)(97+19)}{2(97-19)}=\boxed{58}$ //

15. Challenge: Let $f(x)=\sqrt{ax^2+bx}$. For how many real values "a" is there at least one positive value of "b" for which the domain of "f" and the range of "f" are the same set? AMC 12 2003 #25



case ① $ax^2+bx=0$

$\boxed{a=0}$ ~~$b=0$~~

case ② $ax^2+bx<0$

impossible, no b

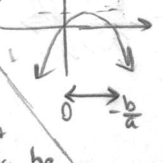
case ③ $ax^2+bx>0$

Roots: $0,-\frac{b}{a}$

if $-\frac{b}{a}$ is $-^{ve}$, then our range cannot match that, so $-\frac{b}{a}$ must be positive.

so a is negative

④ if a is negative ✓



now, our $+^{ve}$ area is between 0 and $-\frac{b}{a}$

OUR RANGE ranges from 0 to MAX now.

max: $\left(\frac{b}{2a},\sqrt{\frac{b^2}{4a}}\right)$

MAX y-value: $\frac{b}{2\sqrt{-a}}$ has to match max x-value...

continued next page

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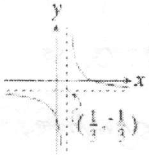
6-6. For what number k can the graph of $y = \frac{x-3}{1-3x}$ be transformed into the graph of $xy = k$ by a translation? [NOTE: If a *translation* moves every point a units horizontally and b units vertically, then the translation transforms the point (x,y) into the point $(x+a, y+b)$.]

Problem 6-6

The graph is known to be a hyperbola with vertical asymptote $x = 1/3$ and horizontal asymptote $y = -1/3$.

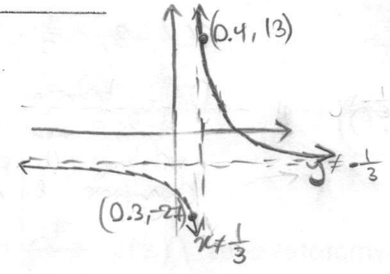


Method I: Look at the equation that results after shifting axes to coincide with the asymptotes, so that $(\frac{1}{3}, -\frac{1}{3})$ is the origin of the translated coordinate system. Then (x,y) of the original coordinate system becomes (x', y') in the new coordinate system, where $x' = x - \frac{1}{3}$ and $y' = y + \frac{1}{3}$. For example, $(x,y) = (\frac{1}{3}, -\frac{1}{3})$ in the original coordinate system becomes $(x', y') = (0,0)$ in the new coordinate system. Using the substitutions $x = x' + \frac{1}{3}$ and $y = y' - \frac{1}{3}$, we get $y' - \frac{1}{3} = \frac{x' + 1/3 - 3}{-3x'}$. Now if we clear fractions and simplify, we'll get $x'y' = \boxed{\frac{8}{9}}$.

Method II: Using long division, $y = -\frac{1}{3} - \frac{8}{3(3x-1)}$, so $y + \frac{1}{3} = \frac{8}{9(x - \frac{1}{3})}$, or $(x - \frac{1}{3})(y + \frac{1}{3}) = \frac{8}{9}$

Method III: Let $(x+a)(y+b) = k$. Then, $y(x+a) = k - bx - ab$, so $y = \frac{k-bx-ab}{x+a}$. But, $y = \frac{x-3}{1-3x} = \frac{-x/3+1}{-1/3+x}$, so $\frac{-x/3+1}{-1/3+x} = \frac{k-bx-ab}{x+a}$. Equate denominators to get $a = -\frac{1}{3}$. Thus, $\frac{-x/3+1}{-1/3+x} = \frac{k-bx+b/3}{x-1/3}$. Now equate numerators to get $k - bx + \frac{b}{3} = -\frac{x}{3} + 1$. Since the coefficients of x must be equal, $-bx = -\frac{x}{3}$, and $b = \frac{1}{3}$. Finally, $k - \frac{x}{3} + \frac{1}{9} = -\frac{x}{3} + 1$, so $k = \frac{8}{9}$.

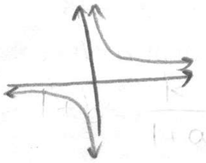
$xy = k \Rightarrow y = \frac{k}{x}$



$(x+a, y+b)$

$a = -\frac{1}{3}$ $b = \frac{1}{3}$

$y = \frac{k}{x}$:



since $(1,1)$ is on $y = \frac{x-3}{1-3x}$, $(\frac{2}{3}, \frac{4}{3})$ must be on $y = \frac{k}{x}$ as it has been transformed accordingly.

$$\frac{4}{3} = \frac{k}{\frac{2}{3}} \Rightarrow k = \frac{4}{3} \times \frac{2}{3} = \boxed{\frac{8}{9}}$$

Question 14 continued...

$$\frac{b}{2\sqrt{-a}} = -\frac{b}{a} \Rightarrow \boxed{a = -4}$$

$\boxed{\text{Two values of } a: -4, 0}$ for which one or less one +ve value of b exists and...

Repeated

~~25. Let $f(x) = \sqrt{ax^2 + bx}$. For how many real values of a is there at least one positive value of b for which the domain of f and the range of f are the same set?~~

~~(A) 0 (B) 1 (C) 2 (D) 3 (E) infinitely many~~

25. (C) The domain of f is $\{x \mid ax^2 + bx \geq 0\}$. If $a = 0$, then for every positive value of b, the domain and range of f are each equal to the interval $[0, \infty)$, so 0 is a possible value of a.
If $a \neq 0$, the graph of $y = ax^2 + bx$ is a parabola with x-intercepts at $x = 0$ and $x = -b/a$. If $a > 0$, the domain of f is $(-\infty, -b/a] \cup [0, \infty)$, but the range of f cannot contain negative numbers. If $a < 0$, the domain of f is $[0, -b/a]$. The maximum value of f occurs halfway between the x-intercepts, at $x = -b/2a$, and

$$f\left(-\frac{b}{2a}\right) = \sqrt{a\left(\frac{b^2}{4a^2}\right) + b\left(-\frac{b}{2a}\right)} = \frac{b}{2\sqrt{-a}}.$$

Hence, the range of f is $[0, b/2\sqrt{-a}]$. For the domain and range to be equal, we must have

cont...

↑

Sol to q.14

$$-\frac{b}{a} = \frac{b}{2\sqrt{-a}} \quad \text{so} \quad 2\sqrt{-a} = -a.$$

The only solution is $a = -4$. Thus there are two possible values of a, and they are $a = 0$ and $a = -4$.

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8. The vertical asymptote and horizontal asymptote for $f(x) = \frac{\sqrt{x}}{x+4}$ are repeased
(a) $x = -4, y = 0$ (b) no vertical asymptote, $y = 0$ (c) no vertical or horizontal asyr
(d) $x = -4$, no horizontal asymptote (e) $x = -4, y = 1$

\\

1 **Find the vertical asymptotes:**

$$f(x) = \frac{2x^3 + 12x^2 + 22x + 12}{x^2 + 6x + 8}$$

repeased

Consider the function: $f(x) = \dfrac{x^2 + x - 6}{2x^2 + 7x + 3}$. If h is the number of horizontal, v the number of vertical, and s the number of slant asymptotes, what is the ordered triple (h,v,s)?

The first three terms of a geometric sequence are: $\sqrt{3}, \sqrt[3]{3}, \sqrt[6]{3}$; what is the next term?

$t_4 = t_3 r = 3^{\frac{1}{6}} \cdot 3^{-\frac{1}{6}} = 3^0 = \boxed{1}$

$\underbrace{3^{\frac{1}{2}}, 3^{\frac{1}{3}}}_{3^{-\frac{1}{6}}}, 3^{\frac{1}{6}}, \ldots \quad r = 3^{-\frac{1}{6}}$

I How many vertical asymptotes does $f(x) = \dfrac{4}{x^2 + 1}$ have?

Read as: How many vertical asymptotes does f of x equals 4 divided by the quantity x squared plus 1 have?

How many integers are in the range of the function $y(x) = \dfrac{4x^2 + 75}{2x^2 + 3}$?

The graph of $y = \frac{3x^3 + x^2 + 4}{x^2 - 48}$ in the Cartesian plane has asymptotes $x = a, x = b$, and $y = cx + d$. Evaluate $abcd$.